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Correlation Functions of Quantum Toroidal $\mathfrak{gl}_1$ Algebra

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Abstract

In this paper, we study the correlation functions of the quantum toroidal $\mathfrak{gl}_1$ algebra. The first key properties we establish are similar to those of the correlation functions of quantum affine algebras $U_q\mathfrak{n}_+$ as established by Enriquez in (Eneiquez, 2000), while the proof of the remaining key "vanishing property" relies on a certain "Master Equality" of formal series, which constitutes the main technical result of this paper.

1. Introduction

The quantum toroidal $\mathfrak{gl}_n$ algebras, where $n > 2$, were introduced more than 20 years ago in (Ginzburg, 1995). However, their representations are not fully understood yet. Surprisingly, the $\mathfrak{gl}_1$ counterpart of those algebras, the quantum toroidal $\mathfrak{gl}_1$, has been introduced later in (Feigin, 2011) and (Miki, 2007) and has attracted lots of attention over the last decade from both mathematicians and physicists due to its close relation to several other topics, which include

- the q-AGT conjecture
- spherical DAHA
- knot invariants
- the Hall algebra of an elliptic curve

Finally, the proper definition of the quantum toroidal algebra of $\mathfrak{gl}_2$ was given only recently in (Feigin, 2011) .

Yet another combinatorial perspective to the quantum toroidal algebras is given via the trigonometric version of the Feigin-Odesskii shuffle algebras of (Feigin, 1997), (Negut, 2013), (Negut, 2014) and (Tsymbaiuk, 2018). In this approach, the elements of the "positive part" of the quantum toroidal algebra are realized as rational symmetric functions subject to rather simple "pole condition" and more interesting "wheel condition," which specify the vanishing of those functions under certain specializations of the variables to a multiple of each other.

An interesting perspective to the aforementioned "wheel" conditions was provided by Enriquez in (Eneiquez, 2000), where he explained how these conditions arise naturally in the consideration of the correlation functions of quantum affine algebras.

The main objective of this paper is to establish similar properties of the correlation functions of the quantum toroidal algebras, thus providing an interesting perspective to the "wheel" conditions in that setup. The case of quantum toroidal algebras is particularly interesting in the setup of $\mathfrak{gl}_n$, since this is the only case when one has two deformation parameters instead of one. Historically, it is common to encode those two parameters via q_1, q_2, q_3 subject to $q_1q_2q_3 = 1$.

However, for $n > 2$, the defining relations for the quantum toroidal $\mathfrak{gl}_n$ algebras look very similar to those of the quantum affine algebras of $\mathfrak{gl}_n$ for $n > 2$. In particular, we can apply the results of (Eneiquez, 2000) to those quantum toroidal algebras. This only leaves the cases $n = 1, 2$ to consider, which is the subject of the current note.

The structure of the paper is as follows. We recall the definition of the positive half of the quantum toroidal $\mathfrak{gl}_1$ algebra in Section 2. We introduce the notion of correlation functions of this algebra in Section 3, and establish its first key properties in Theorems 3.2, 3.3, 3.4. The key "vanishing property" (Theorem 5.1) is established in Section 5 and is essentially based on a particularly interesting equality of formal series, which we refer to as the "Master Equality" in Section 4.

Finally, in Section 6, we present several routes to pursue in the future, which include computing the correlation functions explicitly for different choices of vectors and covectors and establishing similar vanishing properties of the quantum toroidal $\mathfrak{gl}_2$ algebras by proving the conjectured Master Equality in that setup.

2. General Setup

Given any three non-zero complex numbers $q_1, q_2, q_3 \in \mathbb{C}^\times$ such that $q_1q_2q_3 = 1$, the "**positive half**" of the quantum toroidal algebra $U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ defined in (Feigin, 2011) and (Miki, 2007) is an associative algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $\{e_k\}_{k=-\infty}^{+\infty}$ subject to two crucial defining relations (1), (2) listed below.

Definition 2.1. The **quadratic relation** is given by the identity

$$(z - q_1 w)(z - q_2 w)(z - q_3 w)e(z)e(w) = (q_1 z - w)(q_2 z - w)(q_3 z - w)e(w)e(z), \tag{1}$$

where z, w are formal variables and the **current** $e(z)$ is the formal series defined by

$$e(z) = \sum_{k \in \mathbb{Z}} e_k z^{-k}.$$

Note that (1) represents an infinite family of simple relations on the generators. Explicitly, looking at the monomial $z^{-k}w^{-l}$ for some integers k, l, its coefficient on the *LHS* (left-hand side) of (1) is equal to

$$e_{k+3}e_l - (q_1 + q_2 + q_3)e_{k+2}e_{l+1} + (q_1q_2 + q_2q_3 + q_1q_3)e_{k+1}e_{l+2} - e_k e_{l+3},$$

while the coefficient on the *RHS* (right-hand side) of (1) is

$$e_l e_{k+3} - (q_1q_2 + q_2q_3 + q_1q_3)e_{l+1}e_{k+2} + (q_1 + q_2 + q_3)e_{l+2}e_{k+1} - e_{l+3}e_k.$$

Thus (1) is equivalent to a collection of the following relations for any $a, b \in \mathbb{Z}$:

$$(e_{a+3}e_b + e_{b+3}e_a) - (q_1 + q_2 + q_3)(e_{a+2}e_{b+1} + e_{b+2}e_{a+1})+$$
$$(q_1q_2 + q_2q_3 + q_1q_3)(e_{a+1}e_{b+2} + e_{b+1}e_{a+2}) - (e_a e_{b+3} + e_b e_{a+3}) = 0.$$

As we shall see in the later sections, the main effect of the quadratic relation (1) is to help us commute $e(z), e(w)$.

Definition 2.2. Given any function in 3 variables $F(a, b, c)$, its "symmetrization"

$$\sum_{sym\{a,b,c\}} F(a, b, c)$$

is defined via

$$\sum_{sym\{a,b,c\}} F(a, b, c) := F(a, b, c) + F(a, c, b) + F(b, c, a) + F(b, a, c) + F(c, a, b) + F(c, b, a).$$

Definition 2.3. The **cubic relation** given by the identity

$$\sum_{sym\{z_1,z_2,z_3\}} \frac{z_2}{z_3}[e(z_1), [e(z_2), e(z_3)]] = 0. \tag{2}$$

Here, the $[,]$ is a skew-symmetric bi-linear map defined by $[a, b] = ab - ba$. Note that it satisfies the basic properties: $[a, a] = 0$, $[a, b] = -[b, a]$, and $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ (called the Jacobi identity). It could be easily verified that all these are indeed true for our definition of $[,]$ in this case.

Just as before, the relation in (2) invokes infinitely many relations of the generators obtained by comparing the coefficients of monomials $z_1^{-a}z_2^{-b}z_3^{-c}$, where a, b, c are integers. From now on, the cubic relation (2) shall be referred to as the "**Serre**" relation.

Remark. Although we only study the "**positive half**" of the algebra, let us note that the entire quantum toroidal algebra $U_{q_1q_2q_3}(\mathfrak{gl}_1)$ can be recovered as a **Drinfeld double** of $U^+_{q_1q_2q_3}(\mathfrak{gl}_1)$.

The following lemma provides a convenient way to work with the Serre relation:

Lemma 2.1 (Serre relation)**.** *The cubic relation (2) of the quantum toroidal algebra $U^+_{q_1q_2q_3}(\mathfrak{gl}_1)$ is equivalent to*

$$\sum_{sym\{z_1,z_2,z_3\}} e(z_1)e(z_2)e(z_3) \cdot \left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_1}{z_2} - \frac{z_3}{z_2}\right) = 0. \tag{3}$$

Proof. Straightforward computation: replace terms like $[a, b]$ with $ab - ba$ in (2) and rearrange the sums. □

In particular, comparing the coefficients of the monomial $z_1^{-a}z_2^{-b}z_3^{-c}$ in (3), we obtain the equality

$$\sum_{sym\{a,b,c\}} \left(e_a e_{b+1} e_{c-1} + e_{a-1} e_{b+1} e_c - e_{a+1} e_{b-1} e_c - e_a e_{b-1} e_{c+1}\right) = 0.$$

In this paper, we will use the property (3) to establish the crucial claim in Section 5.

There will be a few other series which will be utilized throughout the paper:

Definition 2.4. Given any associative algebra B and a formal variable z, we define several different spaces.

- $B[z]$ denotes the space of all polynomials in z with coefficients in B. Explicitly, it denotes the set

$$\left\{\sum_{k=0}^{n} b_k z^k\right\},$$

  where n is some integer and b_k are elements of B.

- $B[z, z^{-1}]$ denotes the space of all Laurent polynomials in z with coefficients in B. Explicitly, it denotes the set

$$\left\{\sum_{k=-m}^{n} b_k z^k\right\},$$

  where m, n are some integers and b_k are elements of B.

- $B[[z, z^{-1}]]$ denotes the space of all formal series in z with coefficients in B. Explicitly, it denotes the set

$$\left\{\sum_{k\in\mathbb{Z}} b_k z^k\right\},$$

  where b_k are elements of B.

- $B((z))$ denotes the space of all formal series

$$\left\{\sum_{k\in\mathbb{Z}} b_k z^k\right\},$$

  where b_k is an element of B for all k with the property that $b_K = 0$ for small enough $K \ll 0$.

Remark. Note that $B[z], B[z, z^{-1}], B((z))$ are associative algebras with obvious multiplication. However, $B[[z, z^{-1}]]$ is not an algebra since multiplying two series $\sum_{n\in\mathbb{Z}} a_n z^n$ and $\sum_{m\in\mathbb{Z}} b_m z^m$, we end up with the series

$$\sum_{k\in\mathbb{Z}} z^k \cdot \left(\sum_{n+m=k} a_n b_m\right),$$

where $\sum_{n+m=k} a_n b_m$ is an infinite sum. Note that if both series $\sum_{n\in\mathbb{Z}} a_n z^n$ and $\sum_{m\in\mathbb{Z}} b_m z^m$ are in $B((z))$, then $\sum_{n+m=k} a_n b_m$ is well-defined for any k (as only finitely many terms are non-zero) and vanishes when k is small enough. Hence the product of $\sum_{n\in\mathbb{Z}} a_n z^n$ and $\sum_{m\in\mathbb{Z}} b_m z^m$ is well-defined in that case.

The first three definitions can be obviously generalized to the multi-variable case of n variables $z_1, ..., z_n$. Here, we present the multi-variable definition for the last series.

Definition 2.5. Given any associative algebra B,

$$B((z_1))((z_2))...((z_n))$$

is the space of all formal series

$$\sum_{i_1,i_2,...,i_n\in\mathbb{Z}} b_{i_1,i_2,...,i_n} z_1^{i_1} ... z_n^{i_n}$$

whose coefficients are elements of B such that they become zero when i_n is small enough and i_j is small enough for fixed $i_{j+1}, ..., i_n$ for any $1 \leq j < n$.

There are two crucial series (4), (5) that we shall be using in the rest of the paper.

Definition 2.6. For the formal variables x, y, the **delta-function** series $\delta(x, y)$ is defined via

$$\delta(x, y) = \sum_{i \in \mathbb{Z}} x^i y^{-i-1}. \tag{4}$$

Definition 2.7. For the formal variables x, y, the series $\frac{1}{x-y}$ is defined via

$$\frac{1}{x-y} := \frac{1}{x} \sum_{i=0}^{\infty} \left(\frac{y}{x}\right)^i = \sum_{i<0} x^i y^{-i-1}. \tag{5}$$

Note that the series in (5) converges to $\frac{1}{x-y}$ in the region $|y| < |x|$ of $\mathbb{C}^2$.

Remark. We note the following equivalent expression for the delta-function:

$$\delta(x, y) = \frac{1}{x-y} + \frac{1}{y-x}. \tag{6}$$

To illustrate Definition 2.5, let us note that $\frac{1}{z_1 - z_2} \in \mathbb{C}((z_1))((z_2))$ as its coefficients vanish when the exponent on z_2 is less than 0 and when the exponent on z_1 is small enough for a fixed z_2. However, $\frac{1}{z_1 - z_2} \notin \mathbb{C}((z_2))((z_1))$ as for any small integers i, the the series $\frac{1}{z_1 - z_2}$ contains a term where the degree on z_1 is i (namely, $z_2^{-i-1} z_1^i$).

3. Correlation Functions and Their Basic Properties

Definition 3.1 ($\mathbb{Z}$-graded representation)**.** A representation V of a $\mathbb{Z}$-graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A[i]$ is called $\mathbb{Z}$-graded if it can be written as the direct sum of subspaces $V = \bigoplus_{j \in \mathbb{Z}} V[j]$ such that the following condition is satisfied: given any $x \in A[m]$ and $v \in V[n]$, we have $xv \in V[n + m]$.

Let V be a $\mathbb{Z}$-graded representation of the algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, the latter being endowed with the $\mathbb{Z}$-grading via $deg(e_k) = k$. We will assume that $V = \bigoplus_{i \leq N} V_i$ for some integer N, which means that the $\mathbb{Z}$-grading is bounded from above (in particular, all *highest weight* representations satisfy this assumption).

Definition 3.2. Pick a vector $v \in V$ and a covector $\epsilon \in V^*$, where V^* is the dual vector space of V. The "**correlation function**" of the quantum toroidal algebra is defined as

$$f(z_1, ..., z_n) = \langle \epsilon, e(z_1)...e(z_n)v \rangle, \tag{7}$$

where $\langle f, v \rangle$ simply denotes the natural pairing between a covector f and a vector v.

By this definition, the series $f(z_1, ..., z_n)$ of (7) is an element of $\mathbb{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_n, z_n^{-1}]]$.

The goal of this section is to establish some important properties of this function (the corresponding properties of correlation functions of $U_q\mathfrak{n}_+$, where $\mathfrak{n}_+$ is the maximal nilpotent subalgebra of a simple Lie algebra $\mathfrak{g}$, were established by Enriquez in [1]). We will start with the first few simpler properties:

Theorem 3.1.

$$f(z_1, ..., z_n) \in \mathbb{C}((z_1))...((z_n)) \tag{8}$$

Proof. By definition, $f(z_1, z_2, \cdots, z_n)$ can be written as

$$\sum_{i_1, \cdots, i_n \in \mathbb{Z}} \langle \epsilon, e_{i_1} \cdot ... \cdot e_{i_n} v \rangle \cdot z_1^{-i_1} ... z_n^{-i_n}.$$

Let v be an element of V_m for some integer m, then for all $i_n > N - m$, $deg(e_{i_n} v) > N \implies e_{i_n} v = 0$. Now, we claim that fixing $i_n, i_{n-1}, ..., i_k$ for some k, then $\langle \epsilon, e_{i_1} \cdot ... \cdot e_{i_n} v \rangle$ vanishes when i_{k-1} is big enough. This is true because the degree of $e_{i_{k-1}} ... e_{i_n} v$ will eventually exceed N, which implies that it is the zero vector (by an assumption on V). □

To proceed further, we will need a technical lemma about the spaces $\mathbb{C}((z_1))...((z_n))$ of Definition 2.5 :

Lemma 3.2.

$$\bigcap_{\{i_1,i_2,...,i_n\}\in S_n} \mathbb{C}((z_{i_1}))...((z_{i_n})) = \mathbb{C}[[z_1,...,z_n]][z_1^{-1},...,z_n^{-1}], \tag{9}$$

where S_n denote the set of all permutations of $\{1,2,...,n\}$.

Proof. Consider the intersection of the following collection of spaces

$$\{\mathbb{C}((z_2))((z_3))...((z_1)), \mathbb{C}((z_3))((z_4))...((z_2)), ..., \mathbb{C}((z_1))((z_2))...((z_n))\}.$$

It is exactly $\mathbb{C}[[z_1,...,z_n]][z_1^{-1},...,z_n^{-1}]$, because for any series in the intersection, the power of z_i is bounded from below as this series belongs to $\mathbb{C}((z_{i+1}))...((z_i))$. Thus, we know that the desired intersection in Lemma 3.2 must be a subset of $\mathbb{C}[[z_1,...,z_n]][z_1^{-1},...,z_n^{-1}]$. Moreover, it is also clear that any element in $\mathbb{C}[[z_1,...,z_n]][z_1^{-1},...,z_n^{-1}]$ belongs to any $\mathbb{C}((z_{i_1}))...((z_{i_n}))$. Thus, the result follows. □

This result will come in handy when we prove the second property of our correlation function.

Theorem 3.3. *The correlation function $f(z_1,z_2,...,z_n)$ can be written as*

$$\frac{B(z_1,z_2,...,z_n)}{\prod_{i<j}((z_i-q_1z_j)(z_i-q_2z_j)(z_i-q_3z_j))}, \tag{10}$$

where $B(z_1,...,z_n)\in\mathbb{C}[[z_1,...,z_n]][z_1^{-1},...,z_n^{-1}]$.

Proof. We first prove the following claim: given any permutation $\{i_1,i_2,...,i_n\}$ in S_n,

$$\begin{aligned}&\prod_{1\le a<b\le n}(z_a-q_1z_b)(z_a-q_2z_b)(z_a-q_3z_b)f(z_1,z_2,...,z_n) = \\ &\prod_{\substack{a<b\\ i_a<i_b}}(z_a-q_1z_b)(z_a-q_2z_b)(z_a-q_3z_b))\prod_{\substack{a<b\\ i_a>i_b}}(q_1z_a-z_b)(q_2z_a-z_b)(q_3z_a-z_b)f(z_{i_1},...,z_{i_n}).\end{aligned} \tag{11}$$

This directly follows from the quadratic relation (1): given any pairs (a,b) such that $a<b, i_a<i_b$, we do not need to ever commute $e(z_{i_a}), e(z_{i_b})$ from the second expression in order to achieve the first; however, we have to commute every pair $(e(z_{i_a}), e(z_{i_b}))$ such that $i_a>i_b, a<b$ exactly once, at the cost of switching $(z_a-q_1z_b)(z_a-q_2z_b)(z_a-q_3z_b)$ with $(q_1z_a-z_b)(q_2z_a-z_b)(q_3z_a-z_b)$.

Now, it is clear that by Theorem 3.1 and Lemma 3.2, the first expression lies in the intersection of all spaces $\mathbb{C}((z_{i_1}))...((z_{i_n}))$. Thus, it is an element of $\mathbb{C}[[z_1,...,z_n]][z_1^{-1},...,z_n^{-1}]$. Therefore, the conclusion follows. □

Theorem 3.4. *$B(z_1,z_2,...,z_n)$ is a Laurent polynomial.*

Proof. First, we know that the degree of $f(z_1,z_2,...,z_n)$ in each variable is bounded from below. Thus, it suffices to prove that the degree is also bounded from above. Assume that $\epsilon\in V_k^*, v\in V_l$ for some integers $k,l\le N$, so that $\langle\epsilon,v'\rangle\ne 0$ only if $v'\in V_k$.

Recall

$$f(z_1,z_2,...,z_n)=\sum_{i_1,...,i_n\in\mathbb{Z}}\langle\epsilon,(e_{i_1}...e_{i_n})v\rangle\cdot z_1^{-i_1}...z_n^{-i_n}.$$

Note that in order for the monomial $z_1^{-i_1}...z_n^{-i_n}$ to have a nontrivial coefficient, we must have

$$i_1+i_2+...+i_n+l=k. \tag{12}$$

However, we know that $i_1,...,i_n$ are bounded from above. Hence, there are only finitely many solutions to (12) given the boundedness. □

Now we can refine the result of Theorem 3.3:

Lemma 3.5.

$$f(z_1,...,z_n)=\frac{\prod_{i<j}(z_i-z_j)\cdot A(z_1,z_2,...,z_n)}{\prod_{i<j}((z_i-q_1z_j)(z_i-q_2z_j)(z_i-q_3z_j))}, \tag{13}$$

where $A(z_1,...,z_n)$ is a Laurent polynomial.

Proof. When we swap an adjacent pair $(k, k+1)$ in the indices, we have that

$$B(z_1, z_2, ..., z_n) = -B(z_1, ..., z_{k+1}, z_k, ..., z_n). \tag{14}$$

i.e. $B(z_1, z_2, ..., z_n)$ is a skew-symmetric Laurent polynomial.

The reason for this is that once $k, k+1$ are swapped in the indices, we can rewrite

$$f(z_1, z_2, ..., z_k, z_{k+1}, ..., z_n), f(z_1, ..., z_{k+1}, z_k, ...z_n)$$

using the formula in (10). That way, when we multiply the $f(z_1, z_2, ..., z_k, z_{k+1}, ..., z_n)$ by $(z_k - q_1 z_{k+1})(z_k - q_2 z_{k+1})(z_k - q_3 z_{k+1})$ and $f(z_1, ..., z_{k+1}, z_k, ...z_n)$ by $(z_{k+1} - q_1 z_k)(z_{k+1} - q_2 z_k)(z_{k+1} - q_3 z_k)$, we can easily see that the former is equal to the negative of the latter due to the quadratic relation (1), and the identity (14) follows. In other words, whenever we swap any two adjacent indices, we have to negate the value of $B(z_1, z_2, ..., z_n)$. Now, in order to swap any pairs (i, j), we always have to swap $2(j-i)-1$ adjacent indices. Since the number of swaps is always odd, we know that when any pair of indices (i, j) is swapped in $B(z_1, z_2, ..., z_i, ..., z_j, ...z_n)$, the new function $B(z_1, z_2, ..., z_j, ..., z_i, ..., z_n)$ is equivalent to the negative of the original function.

As a skew-symmetric Laurent polynomial, B must be divisible by $\prod_{i<j}(z_i - z_j)$, which implies (13). □

Since $B(z_1, ..., z_n)$ is skew-symmetric as shown above, and the product $\prod_{i<j}(z_i - z_j)$ is also skew-symmetric, we obtain the following:

Corollary 3.5.1. *$A(z_1, z_2, ..., z_n)$ from (13) is a symmetric Laurent polynomial.*

4. The Master Equality

The goal of this section is to establish the following equality of formal series:

Theorem 4.1 (Master Equality)**.** *The following equality of formal series holds:*

$$\sum_{sym\{z_1,z_2,z_3\}} \left(\frac{(z_1 - z_3)(z_2 - z_3)(z_1 - z_2)}{\prod_{i<j}(z_i - q_1 z_j)(z_i - q_2 z_j)(z_i - q_3 z_j)} \left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_3}{z_2} - \frac{z_1}{z_2} \right) \right) =$$
$$(z_3^{-4}) \left(\alpha \sum_{sym\{z_1,z_2,z_3\}} \delta(z_1, q_3 z_2)\delta(z_2, q_1 z_3) + \beta \sum_{sym\{z_1,z_2,z_3\}} \delta(z_1, q_3 z_2)\delta(z_2, q_2 z_3) \right) \tag{15}$$

for constants $\alpha, \beta \in \mathbb{C}^\times$ to be determined at the end of the section (see (24)).

Following Definition 2.7, note that each factor $\frac{1}{z_i - q_k z_j}$ in the LHS of (15) is treated as a formal series

$$\frac{1}{z_i} + q_k \frac{z_j}{z_i^2} + q_k^2 \frac{z_j^2}{z_i^3} + ...,$$

which converges to the rational series $\frac{1}{z_i - q_k z_j}$ in the region $|z_i| > |q_k z_j|$. We shall say that the series $\frac{1}{z_i - q_k z_j}$ is represented by the rational function in the same name in that region.

Using the argument of analytic continuation, it suffices to prove (15) under the following condition:

$$|q_1|, |q_2| < 1. \tag{16}$$

Our key observation is that among the 18 formal series $\frac{1}{z_i - q_k z_j} (i \neq j, 1 \leq i, j, k \leq 3)$ in the LHS of (15), there are 12 (corresponding to the case $k = 1, 2$) which can be represented by the same-named rational functions in a non-empty open region A of $\mathbb{C}^3$ (for example, consider an open neighbourhood of $z_1 = z_2 = z_3 \neq 0$). Thus, we can rewrite the LHS of (15) as

$$\sum_{sym\{z_1,z_2,z_3\}} \left(\prod_{i<j} \left(-\frac{1}{(q_3 z_j - z_i)} + \delta(z_i, q_3 z_j) \right) \frac{(z_1 - z_3)(z_2 - z_3)(z_1 - z_2)}{\prod_{i<j}(z_i - q_1 z_j)(z_i - q_2 z_j)} \left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_3}{z_2} - \frac{z_1}{z_2} \right) \right). \tag{17}$$

Here, we have replaced series of the form $\frac{1}{z_i - q_3 z_j}$ with $\delta(z_i, q_3 z_j) - \frac{1}{q_3 z_j - z_i}$, see remark after Definition 2.7, where $\frac{1}{q_3 z_j - z_i}$ can be represented by the same-named rational function on the region A.

To proceed further, we need to recall the key property of the delta-function:

Lemma 4.2. *For any formal series $F(x) \in \mathbb{C}[[x, x^{-1}]]$, we have $\delta(x, y)F(x) = \delta(x, y)F(y)$.*

Corollary 4.2.1. *In particular, taking $F(x) = x$, we obtain $\delta(x, y)(x - y) = 0$.*

Proof of Lemma 4.2. Straightforward: compare the coefficients of $x^{i+j}y^{-i-1}$ on both sides. □

Applying Corollary 4.2.1 twice, we obtain:

Lemma 4.3. $(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)\delta(z_1, q_3z_2)\delta(z_1, q_3z_3) = 0$

Proof. We obtain after applying Corollary 4.2.1 twice that

$$\begin{aligned} &(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)\delta(z_1, q_3z_2)\delta(z_1, q_3z_3) \\ &= (z_1 - z_2)\left(\frac{z_1}{q_3} - \frac{z_1}{q_3}\right)(z_1 - z_3)\delta(z_1, q_3z_2)\delta(z_1, q_3z_3) \\ &= 0. \end{aligned} \tag{18}$$

□

Thus, opening up the brackets in (17), we obtain the following result:

Corollary 4.3.1. *In the LHS of* (17), *terms with more than one delta-function vanish.*

Thus, only terms with **no** or **just one** delta-function factor remain.

Lemma 4.4 (terms without delta factors vanish)**.** *The following equality of rational functions holds:*

$$\sum_{sym\{z_1,z_2,z_3\}} \left(\frac{(z_1 - z_3)(z_2 - z_3)(z_1 - z_2)}{\prod_{i<j}(z_i - q_1z_j)(z_i - q_2z_j)(q_3z_j - z_i)} \left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_3}{z_2} - \frac{z_1}{z_2}\right)\right) = 0.$$

Proof. This is a straightforward computation verified on Matlab. □

Therefore, we are only left with terms containing exactly 1 delta factor. In other words, the sum in (17) is equal to

$$\sum_{i\neq j} \delta(z_i, q_3z_j)X_{ij} = \sum_{sym\{z_1,z_2,z_3\}} \delta(z_1, q_3z_2) \cdot X_{12}, \tag{19}$$

where X_{12} is a series in z_1, z_2, z_3 explicitly given by $X_{12} = \frac{q_3-1}{(q_3-q_1)(q_3-q_2)} \cdot \frac{1}{z_2} \cdot X^*_{12}$ with

$$\begin{aligned} X^*_{12} = &\frac{(z_1 - z_3)(z_2 - z_3)\left(\frac{1}{q_3} - q_3 + \frac{z_2}{z_3} - \frac{z_3}{z_2}\right)}{(z_1 - q_1z_3)(z_1 - q_2z_3)(q_3z_3 - z_1)(z_2 - q_1z_3)(z_2 - q_2z_3)(q_3z_3 - z_2)} + \\ &\frac{(z_1 - z_3)(z_3 - z_2)\left(\frac{z_3}{z_1} + \frac{z_3}{z_2} - \frac{z_1}{z_3} - \frac{z_2}{z_3}\right)}{(z_1 - q_1z_3)(z_1 - q_2z_3)(q_3z_3 - z_1)(z_3 - q_1z_2)(z_3 - q_2z_2)(q_3z_2 - z_3)} + \\ &\frac{(z_3 - z_1)(z_3 - z_2)\left(-\frac{1}{q_3} + q_3 + \frac{z_1}{z_3} - \frac{z_3}{z_1}\right)}{(z_3 - q_1z_1)(z_3 - q_2z_1)(q_3z_1 - z_3)(z_3 - q_1z_2)(z_3 - q_2z_2)(q_3z_2 - z_3)}. \end{aligned} \tag{20}$$

Using Lemma 4.2, we can switch z_1 with q_3z_2 to get rid of z_1, so that

$$\delta(z_1, q_3z_2)X_{12} = \delta(z_1, q_3z_2)\frac{q_3 - 1}{(q_3 - q_1)(q_3 - q_2)} \cdot \frac{1}{z_2} \cdot X'_{12},$$

with

$$\begin{aligned} X'_{12} = &-\left(\frac{1}{q_3}\right)\frac{(q_3z_2 - z_3)\left(\frac{1}{q_3} - q_3 + \frac{z_2}{z_3} - \frac{z_3}{z_2}\right)}{(q_3z_2 - q_1z_3)(q_3z_2 - q_2z_3)(z_2 - q_1z_3)(z_2 - q_2z_3)(q_3z_3 - z_2)} + \\ &\left(\frac{1}{q_3}\right)\frac{\left(\frac{z_3}{q_3z_2} + \frac{z_3}{z_2} - \frac{q_3z_2}{z_3} - \frac{z_2}{z_3}\right)}{(q_3z_2 - q_1z_3)(q_3z_2 - q_2z_3)(z_3 - q_1z_2)(z_3 - q_2z_2)} - \\ &\left(\frac{z_3 - z_2}{(z_3 - q_1q_3z_2)(z_3 - q_2q_3z_2)}\right)\frac{\left(-\frac{1}{q_3} + q_3 + \frac{q_3z_2}{z_3} - \frac{z_3}{q_3z_2}\right)}{(q_3^2z_2 - z_3)(z_3 - q_1z_2)(z_3 - q_2z_2)}. \end{aligned} \tag{21}$$

Note that

$$\frac{z_3 - z_2}{(z_3 - q_1 q_3 z_2)(z_3 - q_2 q_3 z_2)} = \frac{q_1 q_2}{q_1 - q_2}\left(\frac{q_1 - 1}{q_1 z_3 - z_2} - \frac{q_2 - 1}{q_2 z_3 - z_2}\right).$$

Here, as before, each factor $\frac{1}{az_2 - bz_3}$ is treated as a formal series following Definion 2.7. Evoking the condition (16), we see that all such factors in the formula of X'_{12} can be represented by the same-named rational function on the open neighbourhood $B \in \mathbb{C}^2$ of $z_2 = z_3 \neq 0$ except for $\frac{1}{q_1 z_3 - z_2}, \frac{1}{q_2 z_3 - z_2}$. And again, following the remark after Definition 2.7, we replace $\frac{1}{q_k z_3 - z_2}$ by $\delta(z_2, q_k z_3) - \frac{1}{z_2 - q_k z_3}$, where $\frac{1}{z_2 - q_k z_3}$ is now represented by the same-named rational function on B.

Thus, the third summand in the formula for X'_{12} can be written as follows:

$$\frac{\left(-\frac{1}{q_3} + q_3 + \frac{q_3 z_2}{z_3} - \frac{z_3}{q_3 z_2}\right)}{(q_3^2 z_2 - z_3)(z_3 - q_1 z_2)(z_3 - q_2 z_2)} \times$$
$$\left(\frac{q_1 q_2}{q_1 - q_2}\left((q_1 - 1)\left(-\frac{1}{z_2 - q_1 z_3} + \delta(z_2, q_1 z_3)\right) - (q_2 - 1)\left(-\frac{1}{z_2 - q_2 z_3} + \delta(z_2, q_2 z_3)\right)\right)\right).$$

Now we shall play the same game: open up the bracket and group terms with 1) no delta factor and 2) exactly one delta factor.

Lemma 4.5. *The sum of terms without delta factors (viewed as a rational function in z_2, z_3) is zero.*

Proof. Since each term can be now viewed as a rational function, this becomes a straightforward computation which we verified using Matlab. □

Thus, we are left with terms which contain only one delta factor. In conclusion, the term X'_{12} is equal to

$$\frac{\left(-\frac{1}{q_3} + q_3 + \frac{q_3 z_2}{z_3} - \frac{z_3}{q_3 z_2}\right)}{(q_3^2 z_2 - z_3)(z_3 - q_1 z_2)(z_3 - q_2 z_2)} \cdot \frac{q_1 q_2}{q_1 - q_2} \cdot (q_1 - 1)\delta(z_2, q_1 z_3) -$$
$$\frac{\left(-\frac{1}{q_3} + q_3 + \frac{q_3 z_2}{z_3} - \frac{z_3}{q_3 z_2}\right)}{(q_3^2 z_2 - z_3)(z_3 - q_2 z_2)(z_3 - q_1 z_2)} \cdot \frac{q_1 q_2}{q_1 - q_2} \cdot (q_2 - 1)\delta(z_2, q_2 z_3).$$

After replacing all z_2 terms with $q_1 z_3$, in the first line and $q_2 z_3$ in the second line, we obtain the following formula

$$\begin{aligned}\delta(z_1, q_3 z_2)X_{12} = & z_3^{-4}\frac{q_2}{q_1(q_1 - q_3)(q_1 - q_2)(q_2 - q_3)}\delta(z_1, q_3 z_2)\delta(z_2, q_1 z_3) - \\ & z_3^{-4}\frac{q_1}{q_2(q_1 - q_3)(q_1 - q_2)(q_2 - q_3)}\delta(z_1, q_3 z_2)\delta(z_2, q_2 z_3).\end{aligned} \tag{22}$$

Since the formula in (19) is symmetric, the general form follows.

Theorem 4.6.

$$\begin{aligned}\delta(z_i, q_3 z_j)X_{ij} = & z_k^{-4}\frac{q_2}{q_1(q_1 - q_3)(q_1 - q_2)(q_2 - q_3)}\delta(z_i, q_3 z_j)\delta(z_j, q_1 z_k) - \\ & z_k^{-4}\frac{q_1}{q_2(q_1 - q_3)(q_1 - q_2)(q_2 - q_3)}\delta(z_i, q_3 z_j)\delta(z_j, q_2 z_k),\end{aligned} \tag{23}$$

where $k \in \{1, 2, 3\}$ is determined via $k \neq i, j$.

Corollary 4.6.1.

$$\alpha = \frac{q_2}{q_1(q_1 - q_2)(q_1 - q_3)(q_2 - q_3)}, \beta = -\frac{q_1}{q_2(q_1 - q_2)(q_1 - q_3)(q_2 - q_3)} \tag{24}$$

This completes out proof of Theorem 4.1.

The reason why the Master Equality is so important is that we are going to use it combined with the Serre relation discussed in Section 2 to establish the vanishing property of $A(z_1, z_2, ..., z_n)$ from Section 3.

5. Vanishing Property for Correlation Functions

In this section, we establish the key property of the correlation functions (7). Recall the Laurent polynomial

$$A(z_1, z_2, ..., z_n)$$

from (13).

Theorem 5.1 (Vanishing Property). $A(z_1, ..., z_n)$ *vanishes when the following condition is met:*

$$\left\{\frac{z_1}{z_2}, \frac{z_2}{z_3}, \frac{z_3}{z_1}\right\} = \left\{q_1, q_2, q_3\right\}.$$

Remark. Note that $A(z_1, z_2, ..., z_n)$ is symmetric, hence, we can replace $\{z_1, z_2, z_3\}$ with $\{z_a, z_b, z_c\}$ such that $a \neq b, b \neq c\ c \neq a$, and the theorem still holds.

First, we consider a result that follows from the Serre relation:

Lemma 5.2.

$$\sum_{sym\{z_1,z_2,z_3\}} f(z_1, z_2, z_3, z_4, ..., z_n)\left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_1}{z_2} - \frac{z_3}{z_2}\right) = 0 \tag{25}$$

Proof. The above *LHS* equals

$$\epsilon\left(\sum_{sym\{z_1,z_2,z_3\}} e(z_1)e(z_2)e(z_3)\left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_1}{z_2} - \frac{z_3}{z_2}\right)e(z_4)...e(z_n)v\right),$$

and the expression in the large bracket is zero, due to the Serre relation (3). □

Now, we examine an equivalent expression to the *LHS* of (25) :

$$\sum_{sym\{z_1,z_2,z_3\}} \frac{\prod_{1\le i<j\le n}(z_i - z_j) \cdot A(z_1, z_2, ..., z_n)}{\prod_{1\le i<j\le n}(z_i - q_1 z_j)(z_i - q_2 z_j)(z_i - q_3 z_j)}\left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_3}{z_2} - \frac{z_1}{z_2}\right). \tag{26}$$

By Lemma 5.2, we know that this is also equal to 0. Note that whenever two variables are swapped in $A(z_1, z_2, ..., z_n)$, the function does not change as it is symmtric. Therefore, the expression (26) is equal to

$$A(z_1, z_2, ..., z_n) \sum_{sym\{z_1,z_2,z_3\}} \frac{a \cdot \prod_{1\le i<j\le 3}(z_i - z_j)}{\prod_{1\le i<j\le 3}(z_i - q_1 z_j)(z_i - q_2 z_j)(z_i - q_3 z_j)}\left(\frac{z_2}{z_3} + \frac{z_2}{z_1} - \frac{z_3}{z_2} - \frac{z_1}{z_2}\right),$$

where $a = \prod_{1\le i<j\le n}^{j>3} \frac{z_i - z_j}{(z_i - q_1 z_j)(z_i - q_2 z_j)(z_i - q_3 z_j)}$.

Thus by the Master Equality (15), we obtain the following equality:

$$A(z_1, z_2, ..., z_n) \cdot a \cdot \left(\alpha \sum_{sym\{z_1,z_2,z_3\}} \delta(z_1, q_3 z_2)\delta(z_2, q_1 z_3) + \beta \sum_{sym\{z_1,z_2,z_3\}} \delta(z_1, q_3 z_2)\delta(z_2, q_2 z_3)\right) = 0. \tag{27}$$

with α, β determined in (24).

Theorem 5.3. *The set of products of delta factors in* (27) *is linearly independent.*

Proof. Note that for any $i, j, k \in \{1, 2, 3\}$ with $i \neq j, i \neq k, j \neq k$, $\delta(z_i, q_3 z_j)\delta(z_j, q_1 z_k)$ is equal to

$$\gamma_{ijk}\delta(z_1, \alpha_{ijk} z_2)\delta(z_2, \beta_{ijk} z_3)$$

for some non-zero constants $\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}$. For example,

$$\delta(z_1, q_3 z_3)\delta(z_3, q_1 z_2) = q_1^{-1} \cdot \delta(z_1, q_3 q_1 z_2)\delta(z_2, q_1^{-1} z_3).$$

In the same manner, $\delta(z_i, q_3 z_j)\delta(z_j, q_2 z_k) = \delta(z_1, \alpha'_{ijk} z_2)\delta(z_2, \beta'_{ijk} z_3)\gamma'_{ijk}$ for non-zero constants $\alpha'_{ijk}, \beta'_{ijk}, \gamma'_{ijk}$. A routine evaluation of the coefficients $\alpha_{ijk}, \beta_{ijk}, \alpha'_{ijk}, \beta'_{ijk}$ shows that all pairs of $(\alpha'_{ijk}, \beta'_{ijk})$ and $(\alpha_{ijk}, \beta_{ijk})$ are pairwise distinct. Therefore, it suffices to establish the following result: the set of products $\{\delta(z_1, A_i z_2)\delta(z_2, B_i z_3)\}_{1\le i\le 12}$ (6 terms obtained via arranging products of the form $\delta(z_i, q_3 z_j)\delta(z_j, q_2 z_k)$ and 6 other terms obtained via aranging products of the form $\delta(z_i, q_3 z_j)\delta(z_j, q_1 z_k)$) is linearly independent given all pairs (A_i, B_i) are pairwise distinct.

Let us expand $\delta(z_1, A_i z_2)\delta(z_2, B_i z_3)$ explicitly as follows:

$$\delta(z_1, A_i z_2)\delta(z_2, B_i z_3) = \delta(z_1, A_i B_i z_3)\delta(z_2, B_i z_3) \tag{28}$$

$$= \sum_{a,b\in\mathbb{Z}} z_1^{-a-1}(A_i B_i z_3)^a \cdot z_2^{-b-1}(B_i z_3)^b \tag{29}$$

$$= \sum_{a,b\in\mathbb{Z}} z_1^{-a-1} z_2^{-b-1} z_3^{a+b}(A_i B_i)^a B_i^b. \tag{30}$$

And assume that $\sum_i c_i \delta(z_1, A_i z_2)\delta(z_2, B_i z_3) = 0$ for some set of constants $\{c_i\}$. We shall start by fixing a and varying b. In that way, when looking at the coefficient on $z_1^{-a-1} z_2^{-b-1} z_3^{a+b}$, we obtain that for any $b \in \mathbb{Z}$,

$$\sum_i c_i (A_i B_i)^a B_i^b = 0.$$

Since a is fixed, we can let $d_i = c_i \cdot (A_i B_i)^a$. In particular, we have $\sum_{i=1}^{12} d_i B_i^k = 0$ for $k = 0, 1, ..., 11$.

• If all B_i are pairwise distinct, then by the Vandermonde determinant formula, this system admits only the trivial solution, which immediately implies that all $c_i = 0$ as $A_i, B_i \neq 0$.

• Now let's look at the case where $B_i = B_j$ for some i, j. In that scenario, without loss of generality we shall assume that $B_1 = B_2 = ... = B_N$ and $B_i \neq B_1$ for $i > N$. Then, applying the Vandermonde formula again implies that $\sum_{i=1}^N c_i (A_i B_i)^a = 0$ for all a, hence $\sum_{i=1}^N c_i A_i^a = 0$. When we vary a and use the Vandermonde determinant formula once again, we likewise get $c_i = 0$ for $i = 1, 2, ..., N$ unless $A_i = A_j$ for some distinct $i, j \in \{1, 2, ..., N\}$. However, the latter can not happen since the pairs (A_i, B_i) are pairwise distinct and $B_i = B_j$ for $i, j \in \{1, 2, ..., N\}$ □

Thus, $A(z_1, z_2, ..., z_n)$ must equal 0 upon the substitution $z_2 = q_1 z_3, z_3 = q_2 z_1$, which concludes the proof of the vanishing property, Theorem 5.1.

Conclusion. Basically, we have shown that all the correlation functions have the form

$$f(z_1, z_2, ..., z_n) = \frac{\prod_{i<j}(z_i - z_j) \cdot A(z_1, z_2, ..., z_n)}{\prod_{i<j}((z_i - q_1 z_j)(z_i - q_2 z_j)(z_i - q_3 z_j))}$$

for some symmetric Laurent polynomial $A(z_1, z_2, ..., z_n)$, which vanish whenever there are three variables z_a, z_b, z_c among the set of n variables $\{z_1, ..., z_n\}$ satisfying the property

$$\left\{\frac{z_1}{z_2}, \frac{z_2}{z_3}, \frac{z_3}{z_1}\right\} = \left\{q_1, q_2, q_3\right\}.$$

6. Future Work

In the previous sections, we have established some crucial properties of the correlation functions of $U_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. In this section, we primarily outline two directions we plan to pursue in the future.

- Compute explicitly some of the correlation functions of the quantum toroidal $\mathfrak{gl}_1$ algebra.
- Generalize the properties of correlation functions of the quantum toroidal $\mathfrak{gl}_1$ to quantum toroidal $\mathfrak{gl}_n$.

6.1 Explicit Computations of Correlation Functions

We shall look at the first point to begin. In order to compute the correlation functions, we need to start with some representations of the algebra. Moreover, this representation has to satisfy the assumptions of Section 3 (namely, to be $\mathbb{Z}$-graded with the $\mathbb{Z}$-grading bounded from above). A basic example of such representation is the **Fock** representation of [3].

Remark. In fact, the Cartan-extended version of $U^{+}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ is a Hopf algebra, thus one can consider the tensor products and duals of basic representations of $U^{+}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ in order to construct a plethora of representations to compute the correlation functions. In this section, we only look at the Fock representation for simplicity.

Definition 6.1. The"**oscillator**" or "**Heisenberg**" algebra $\mathfrak{h}$ is the associative $\mathbb{C}$-algebra generated by $\{a_n\}_{n\in\mathbb{Z}^\times}$ subject to the following relations:

- $[a_m, a_n] = 0$ if $m + n \neq 0$.
- $[a_m, a_{-m}] = m \cdot \frac{1-q_1^{|m|}}{1-q_2^{-|m|}}$.

Note that for any $m \in \mathbb{Z}_{>0}$ the subalgebra generated by $\{a_m, a_{-m}\}$ is isomorphic to the **Weyl** algebra, and these Weyl algebras actually pairwise commute for various m.

Recall the construction of the **Fock module**.

Lemma 6.1. *There is an action of* $\mathfrak{h}$ *on* $F = \mathbb{C}[a_{-1}, a_{-2}, ...]$ *with* a_{-k} *acting as multiplication by* a_{-k} *and* a_k *acting via* $k \cdot \frac{1-q_1^k}{1-q_2^{-k}}\partial_{a_{-k}}$ *for* $k > 0$.

It turns out that F carries a natural action of $U^{+}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ as established in [3] :

Proposition 6.1.1. *The assignment*

$$e(z) \mapsto exp\left(\sum_{n>0}\frac{1-q_2^n}{n}a_{-n}z^n\right)exp\left(-\sum_{n>0}\frac{1-q_2^{-n}}{n}a_n z^{-n}\right)$$

defines an action of $U^{+}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ *on* F. *This representation of* $U^{+}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ *is called the* **Fock module**.

Remark. Here,

$$exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + ...$$

Note that F is $\mathbb{Z}$-graded via $deg(a_k) = k$ with the $\mathbb{Z}-$grading bounded from above by 0. Therefore, $exp\left(-\sum_{n>0}\frac{1-q_2^{-n}}{n}a_n z^{-n}\right)v$ is a finite sum due to the boundedness of the $\mathbb{Z}$-grading. Second, $exp\left(\sum_{n>0}\frac{1-q_2^n}{n}a_{-n}z^n\right)v$ is an infinite sum, but the coefficient of each z^{-k} is a finite sum because of the formula in the remark above.

Let us now compute the correlation functions for $v = 1 \in F$ and $\epsilon = 1^* \in F^*$ where $1^*(1) = 1$ and $1^*(a_{-k}) = 1^*(a_{-k}a_{-m})... = 0$ (in other words, it pairs trivially with terms of positive degree).

In the case $n = 2$, we have that $f(z_1, z_2) = \langle 1^*, e(z_1)e(z_2)1\rangle$ and

$$\begin{aligned} e(z_1)e(z_2) =& exp\left(\sum_{n>0}\frac{1-q_2^n}{n}a_{-n}z_1^n\right)exp\left(-\sum_{n>0}\frac{1-q_2^{-n}}{n}a_n z_1^{-n}\right)\times \\ & exp\left(\sum_{m>0}\frac{1-q_2^m}{m}a_{-m}z_2^m\right)exp\left(-\sum_{n>0}\frac{1-q_2^{-m}}{m}a_m z_2^{-m}\right). \end{aligned} \tag{31}$$

Note that given two endomorphisms A, B of the same vector space V, which both commute with the commutator $[A, B]$, we have

$$e^A \cdot e^B = e^B \cdot e^A \cdot e^{[A,B]}$$

due to the BakerCCampbellCHausdorff formula. Thus the product (31) equals

$$\begin{aligned} & exp\left(\sum_{n>0}\frac{1-q_2^n}{n}a_{-n}(z_1^n + z_2^n)\right)exp\left(-\sum_{n>0}\frac{1-q_2^{-n}}{n}a_n(z_1^{-n} + z_2^{-n})\right)\times \\ & exp\left(-\sum_{n>0}\frac{1-q_2^{-n}}{n}\cdot\frac{1-q_2^n}{n}\cdot n\cdot\frac{1-q_1^n}{1-q_2^{-n}}\cdot\left(\frac{z_2}{z_1}\right)^n\right). \end{aligned}$$

It is clear that $exp(\sum c_n a_n)1 = 1$ for any constants c_n because a_n acts via derivation. Moreover,

$$\langle 1^*, exp(\sum c_n a_{-n})1\rangle = 1$$

as 1^* pairs trivially with terms of positive degree. Thus, the upshot is that

$$f(z_1, z_2) = exp\left(-\sum_{n>0} \frac{(1-q_1^n)(1-q_2^n)}{n} \cdot \left(\frac{z_2}{z_1}\right)^n\right).$$

After expanding the inner product and applying the formula

$$log(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - ...,$$

we obtain the following result:

Proposition 6.1.2. *For $v = 1 \in F, \epsilon = 1^* \in F^*$ and $n = 2$, we have*

$$f(z_1, z_2) = \frac{(1-\frac{z_2}{z_1})(1-q_1q_2\frac{z_2}{z_1})}{(1-q_1\frac{z_2}{z_1})(1-q_2\frac{z_2}{z_1})}$$

Applying the same argument in the case $n > 2$, we arrive at the following result:

Proposition 6.1.3. *For $v = 1 \in F, \epsilon = 1^* \in F^*$ and any $n \in \mathbb{Z}_{>0}$, we have*

$$f(z_1, z_2, ..., z_n) = \prod_{1\leq i<j\leq n} \frac{(1-\frac{z_j}{z_i})(1-q_1q_2\frac{z_j}{z_i})}{(1-q_1\frac{z_j}{z_i})(1-q_2\frac{z_j}{z_i})}$$

In the future, we plan to find explicit correlation functions for other choices of $v \in F$ and $\epsilon \in F^*$.

6.2 Generalization to Quantum Toroidal $\mathfrak{gl}_n$

The second route of the future work is to generalize the key results of correlation functions associated with $\mathfrak{gl}_1$ to $\mathfrak{gl}_n$, where the family of quantum toroidal algebras of $\mathfrak{gl}_n$ is defined in [4] . We shall first look at the case when $n = 2$, which is of primary interest, since the case of $n > 2$ is similar to quantum affine algebras (see remark at the end of this section).

The algebra $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_2)$ is generated by $\{e_{i,n}\}^{n\in\mathbb{Z}}_{i=1,2}$ subject to the defining relations (32), (33), (34) specified below.

Definition 6.2. The quadratic relations are given by

$$(z - q_2w)e_i(z)e_i(w) = -(w - q_2z)e_i(w)e_i(z), \quad i = 1, 2, \tag{32}$$

$$(z - q_1w)(z - q_3w)e_i(z)e_j(w) = (w - q_1z)(w - q_3z)e_j(w)e_i(z), \quad i \neq j, \tag{33}$$

while the Serre relation is given by

$$\begin{aligned}\sum_{sym\{z_1,z_2,z_3\}} \Big(&e_i(z_1)e_i(z_2)e_i(z_3)e_j(w) - (q_2^{-1} + 1 + q_2)e_i(z_1)e_i(z_2)e_j(w)e_i(z_3) + \\ &(q_2^{-1} + 1 + q_2)e_i(z_1)e_j(w)e_i(z_2)e_i(z_3) - e_j(w)e_i(z_1)e_i(z_2)e_i(z_3)\Big) = 0, \quad i \neq j.\end{aligned} \tag{34}$$

For simplicity, we consider the particular ordering such that all e_1-currents appear to the left of e_2-currents. Namely,

$$f(z_1, ..., z_n; w_1, ..., w_m) = \langle\epsilon, e_1(z_1)...e_1(z_n)e_2(w_1)...e_2(w_m)v\rangle.$$

Now, applying similar reasoning as in Section 3, it is easy to show that

$$\begin{aligned}&f(z_1, ..., z_n; w_1, ..., w_m) = \\ &A(z_1, ..., z_n; w_1, ..., w_m) \cdot \frac{\prod_{i<j}(z_i - z_j) \cdot \prod_{i<j}(w_i - w_j)}{\prod_{i<j}(z_i - q_2z_j)\prod_{i<j}(w_i - q_2w_j)\prod_{i\leq n}^{j\leq m}(z_i - q_1w_j)(z_i - q_3w_j)},\end{aligned}$$

where A is a Laurent polynomial symmetric in $\{z_i\}^n_{i=1}$ and $\{w_j\}^m_{j=1}$.

Then, we predict the vanishing property of the quantum toroidal $\mathfrak{gl}_2$ to be the following:

Conjecture 1. *$A(z_1, ..., z_n; w_1, ..., w_m) = 0$ if one of the conditions are met:*

• *There are $\{z_a, z_b, w_k\}$ such that*

$$w_k = q_3 z_a, z_a = q_2 z_b$$

or

$$w_k = q_3^{-1} z_a, z_a = q_2^{-1} z_b,$$

for some $k \in \{1, ..., m\}$ and $a, b \in \{1, ..., n\}$ with $a \neq b$.

• *There are $\{w_a, w_b, z_k\}$ such that*

$$z_k = q_3 w_a, w_a = q_2 w_b$$

or

$$z_k = q_3^{-1} w_a, w_a = q_2^{-1} w_b,$$

for some $a, b \in \{1, ..., m\}, a \neq b$ and $k \in \{1, ..., n\}$

This result will follow directly from the following conjectured Master Equality in the same way Theorem 5.1 was deduced from Theorem 4.1.

Conjecture 2 (Master Equality for the quantum toroidal $\mathfrak{gl}_2$). *The following equality of formal series holds:*

$$\sum_{sym\{z_1,z_2,z_3\}} \prod_{1\le i<j\le 3} \frac{z_i - z_j}{z_i - q_2 z_j} \times \left(\frac{1}{\lambda(z_1, w)\lambda(z_2, w)\lambda(z_3, w)} - \frac{q_2 + 1 + q_2^{-1}}{\lambda(z_1, w)\lambda(z_2, w)\lambda(w, z_3)} + \frac{q_2 + 1 + q_2^{-1}}{\lambda(z_1, w)\lambda(w, z_2)\lambda(w, z_3)} - \frac{1}{\lambda(w, z_1)\lambda(w, z_2)\lambda(w, z_3)} \right) = \sum_{sym\{z_1,z_2,z_3\}} \left(\alpha(z_2, z_3)\delta(w, q_3^{-1} z_1)\delta(z_1, q_2^{-1} z_2) + \beta(z_2, z_3)\delta(w, q_3 z_1)\delta(z_1, q_2 z_2) \right), \tag{35}$$

where $\frac{1}{\lambda(x,y)} = \frac{1}{(x-q_1 y)} \frac{1}{(x-q_3 y)}$ (with both factors viewed as formal series from Definition 2.7) and $\alpha(x, y), \beta(x, y)$ being some non-zero formal series in x, y.

Remark. For $n \geq 3$, the cubic Serre relation of $U^+_{q_1,q_2,q_3}(\mathfrak{gl}_n)$ looks exactly the same as the one for the quantum affine algebras of $\mathfrak{gl}_n$, hence the vanishing property follows directly from [1].

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